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Applications of variational iteration and homotopy perturbation methods to obtain exact solutions for time-fractional diffusion-wave equations

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Abstract

Purpose – The purpose of this paper is to consider the time-fractional diffusion-wave equation. The time-fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0, 2]$. The fractional derivatives are described in the Caputo sense.

Design/methodology/approach – The two methods in applied mathematics can be used as alternative methods for obtaining an analytic and approximate solution for different types of differential equations.

Findings – Four examples are presented to show the application of the present techniques. In these schemes, the solution takes the form of a convergent series with easily computable components. The present methods perform extremely well in terms of efficiency and simplicity.

Originality/value – In this paper, the variational iteration and homotopy perturbation methods are used to obtain a solution of a fractional diffusion equation.

Keywords Differential equations, Iterative methods, Flow, Heat, Numerical analysis

Paper type Research paper

1. Introduction

A fractional diffusion-wave equation is a linear integro partial differential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative term by a fractional derivative of order $\alpha \in (0, 2]$. These equations arise in anomalous diffusion and sub-diffusion systems, the description of fractional random walk and the unification of diffusion and wave propagation phenomena. The nature of the diffusion is characterized by the temporal scaling of the mean-square displacement $\langle r^2(t) \rangle \approx t^{\alpha}$. For standard diffusion $\alpha = 1$, whereas in anomalous sub-diffusion $\alpha < 1$, and in anomalous super-diffusion $\alpha > 1$. Both types of anomalous diffusion have been unified in continuous time random walk models with spatial and temporal memories, see e.g. the reviews in Mainardi (1997), Agrawal (2002), Metzler *et al.* (1999), Metzler and Klafter (2000), Schneider and Wyss (1989), and references therein. Most equations for heat and fluid flow can be described by differential equations, but heat in, e.g., hierarchical wool fibers or bamboos can be described using fractional differential equations, and the hierarchical wool fibers, for example, behave with high heat conduction efficiency (Fan *et al.*, 2008; Zhou *et al.*, 2009).

The solution of a fractional differential equation is much involved. In general, there exists no method that yields an exact solution for a fractional differential equation. No

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analytical method was available before 1998 for such equations even for linear fractional differential equations. In 1998, the variational iteration method (VIM) was first proposed to solve fractional differential equations with greatest success (He, 1998). Many authors found VIM is an effective way to solving fractional equations, both linear and nonlinear (Odibat and Momani, 2006; Das, 2008). Momani and Odibat (2007) and Ganji *et al.* (2008) applied the homotopy perturbation method (HPM) to fractional differential equations and revealed that the HPM is an alternative analytical method for fractional differential equations. Momani *et al.* (2008) and Odibat and Momani (2008) compared solution procedure between VIM and HPM. Recently, Ray (2007) used Adomian decomposition method to obtain exact solutions for time-fractional diffusion-wave equations.

In this paper, we shall consider the time-fractional diffusion equation (Mainardi, 1997):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{1.1}$$

where $\partial^\alpha(\bullet)/\partial t^\alpha$ is the Caputo derivative of order α. In this paper, we use the variational iteration (He and Wu, 2007; He, 2007, 1999a, 2000a; Öziş and Yıldırım, 2007; Yıldırım and Öziş, 2009; Yıldırım, 2008a, b; Labidi and Omrani, 2009) and HPM (He, 1999b, 2000b; Yıldırım and Öziş, 2007; Yıldırım, 2008c-f, 2009a, b; Dehghan and Shakeri, 2008, 2007; Shakeri and Dehghan, 2007, 2008; Achouri and Omrani, 2009; Ghanmi *et al.*, 2009) to obtain a solution of a fractional diffusion equation (1.1).

The solution procedure using He's polynomials in both VIM and HPM were introduced (Mohyud-Din *et al.*, 2009; Noor and Mohyud-Din, 2008). The most development of VIM and HPM were summarized in He (2006a, b, 2008a, b).

2. Basic definitions of fractional calculus

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1

A real function, $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(>\mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \mathrm{d}t, \quad \alpha > 0, \quad x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in Miller and Ross (1993), Samko *et al.* (1993), and Oldham and Spanier (1974); we mention only the following, for $f \in C_\mu,\ \mu \geq -1,\ \ \alpha, \beta \geq 0$, and $\gamma > -1$:

- $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
- $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$, and
- $J^\alpha x^\gamma = \Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1) x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by Caputo in his work on the theory of viscoelasticity (Luchko and Gorneflo, 1998).

Definition 2.3
The fractional derivative *f*(*x*) in the Caputo sense is defined as:

$$D^{\alpha}f(x) = J^{m-\alpha}D^{m}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt, \tag{2.1}$$

for $m-1 < \alpha \leq m,\ m \in N,\ x > 0,\ f \in C_{-1}^{m}$.

Also, we need here two of its basic properties.

Lemma 2.1. If $m-1 < \alpha \leq m,\ m \in N$ and $f \in C_{\mu}^{m},\ \mu \geq -1$, then

$$D^{\alpha}J^{\alpha}f(x) = f(x),$$

and

$$J^{\alpha}D^{\alpha}f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^{+})\frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivatives are considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the foam drainage equation with time- and space-fractional derivatives, and the fractional derivatives are taken in Caputo sense as follows.

Definition 2.4
For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as:

$$D_t^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial t^m}d\tau, & \text{for } m-1 < \alpha < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in N. \end{cases} \tag{2.2}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

3. The HPM

Consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3.1}$$

with boundary conditions:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{3.2}$$

where A is a general differential operator, B is a boundary operator, *f*(*r*) is a known analytic function, and Γ is the boundary of the domain Ω.

The operator A can, generally speaking, be divided into two parts L and N, where L is linear and N is nonlinear, therefore Equation (3.1) can be written as:

$$L(u) + N(u) - f(r) = 0. \tag{3.3}$$

By using homotopy technique, one can construct a homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow \Re$, which satisfies:

$$\mathrm{H}(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{3.4a}$$

or

$$\mathrm{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{3.4b}$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is the initial approximation of Equation (3.1) which satisfies the boundary conditions. Clearly, we have:

$$\mathrm{H}(v, 0) = L(v) - L(u_0) = 0 \tag{3.5}$$

or

$$\mathrm{H}(v, 1) = A(v) - f(r) = 0, \tag{3.6}$$

the changing process of p from zero to unity is just that of $v(r, p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If the embedding parameter $p(0 \leq p \leq 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of Equation (3.4) can be given as a power series in p, i.e.:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{3.7}$$

and setting $p = 1$ results in the approximate solution of Equation (3.1) as:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \cdots \tag{3.8}$$

4. The VIM

The VIM was proposed by He and Wu (2007) and He (1999a, 2000a, 2007), where correction functional for the Equation (3.3) can be written as:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)(L(u_n(\tau)) + N(\tilde{u}_n(\tau)) - f(\tau))d\tau, \tag{4.1}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory (He, 2004), the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

It is obvious now that the main steps of the VIM require first the determination of the Lagrangian multiplier λ that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution:

$$u = \lim_{n \rightarrow \infty} u_n. \tag{4.2}$$

5. Implementation of the present methods

5.1 Applications of the HPM

Example 1. We first consider:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{5.1}$$

Let us consider initial conditions:

$$u(x,0)=\sin(x), \quad u_t(x,0)=0, \tag{5.2}$$

$$0<x<\pi \quad \text{for} \quad 1<\alpha\leq 2, \tag{5.3}$$

and boundary conditions:

$$u(0,t)=u(\pi,t)=0, \quad t\geq 0. \tag{5.4}$$

According to HPM, we readily construct the homotopy:

$$D_t^\alpha u(x,t)=p\frac{\partial^2}{\partial x^2}u(x,t). \tag{5.5}$$

Assume the solution of equation in the form:

$$u=u_0+pu_1+p^2u_2+p^3u_3+p^4u_4+\cdots, \tag{5.6}$$

Substituting (5.6) into Equation (5.5) and collecting terms of the same power of p gives:

$$p^0: D_t^\alpha u_0=0, \quad u(x,0)=u_0(x), \tag{5.7}$$

$$p^1: D_t^\alpha u_1=\frac{\partial^2}{\partial x^2}u_0, \quad u_1(0,t)=u_1(\pi,t)=0, \tag{5.8}$$

$$p^2: D_t^\alpha u_2=\frac{\partial^2}{\partial x^2}u_1, \quad u_2(0,t)=u_2(\pi,t)=0, \tag{5.9}$$

$$p^3: D_t^\alpha u_3=\frac{\partial^2}{\partial x^2}u_2, \quad u_3(0,t)=u_3(\pi,t)=0, \tag{5.10}$$

$$p^4: D_t^\alpha u_4=\frac{\partial^2}{\partial x^2}u_3, \quad u_4(0,t)=u_4(\pi,t)=0, \tag{5.11}$$

$$\ldots$$

Solving the above equations, we obtain:

$$u_0(x,t)=\sin(x), \tag{5.12}$$

$$u_1(x,t)=\frac{-t^\alpha}{\Gamma(\alpha+1)}\sin(x), \tag{5.13}$$

$$u_2(x,t) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x), \tag{5.14}$$

$$u_3(x,t) = \frac{-t^{3\alpha}}{\Gamma(3\alpha+1)}\sin(x), \tag{5.15}$$

$$u_4(x,t) = \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\sin(x), \tag{5.16}$$

$$\ldots$$

and so on. Then, setting $p = 1$ results in the approximate solution of the equation:

$$u = \lim_{p\to 1} u = u_0 + pu_1 + p^2u_2 + p^3u_3 + p^4u_4 + \cdots. \tag{5.17}$$

Therefore, the solution is:

$$u(x,t) = \sum_{r=0}^{\infty}\frac{(-t^{\alpha})^r}{\Gamma(r\alpha+1)}\sin(x), \tag{5.18}$$

which is the exact solution of the problem. The solution can be verified through substitution in equation.

Example 2. We now handle:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{5.19}$$

with initial conditions:

$$u(x,0) = f(x), \quad 0 < x < 2, \tag{5.20}$$

$$u_t(x,0) = 0, \quad 0 < x < 2 \quad \text{for } 1 < \alpha \le 2, \tag{5.21}$$

where

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \end{cases}, \tag{5.22}$$

and boundary conditions:

$$u(0,t) = u(2,t) = 0. \tag{5.23}$$

We see that $f(x)$ is a periodic function with period 2. The Fourier sine series of $f(x)$ in [0, 2] can be obtained as:

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\sin\left(\frac{(2n-1)\pi x}{2}\right), \tag{5.24}$$

because of the fact that Fourier sine series well adapted to functions which are zero at the end points $x = 0$ and $x = 2$ of the interval [0, 2], since all the basis functions $\sin((2n - 1)\pi x/2)$ have this property.

We will then obtain:

$$u_0 = u(x,0) + tu_t(x,0),$$
$$= \sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\sin\left(\frac{(2n-1)\pi x}{2}\right). \tag{5.25}$$

According to HPM, we readily construct the homotopy:

$$D_t^{\alpha}u(x,t) = p\frac{\partial^2}{\partial x^2}u(x,t). \tag{5.26}$$

Substituting (5.6) into Equation (5.5) and collecting terms of the same power of p gives:

$$p^0 : D_t^{\alpha}u_0 = 0, \quad u(x,0) = u_0(x), \tag{5.27}$$

$$p^1 : D_t^{\alpha}u_1 = \frac{\partial^2}{\partial x^2}u_0, \quad u_1(0,t) = u_1(2,t) = 0, \tag{5.28}$$

$$p^2 : D_t^{\alpha}u_2 = \frac{\partial^2}{\partial x^2}u_1, \quad u_2(0,t) = u_2(2,t) = 0, \tag{5.29}$$

$$p^3 : D_t^{\alpha}u_3 = \frac{\partial^2}{\partial x^2}u_2, \quad u_3(0,t) = u_3(2,t) = 0, \tag{5.30}$$

$$p^4 : D_t^{\alpha}u_4 = \frac{\partial^2}{\partial x^2}u_3, \quad u_4(0,t) = u_4(2,t) = 0, \tag{5.31}$$

$$\ldots$$

Solving the above equations, we obtain:

$$u_0 = \sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\sin\left(\frac{(2n-1)\pi x}{2}\right), \tag{5.32}$$

$$u_1(x,t) = \frac{-t^{\alpha}}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right] \times \left(\frac{(2n-1)\pi}{2}\right)^2\sin\left(\frac{(2n-1)\pi x}{2}\right), \tag{5.33}$$

$$u_2(x,t) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right] \times \left(\frac{(2n-1)\pi}{2}\right)^4\sin\left(\frac{(2n-1)\pi x}{2}\right), \tag{5.34}$$

$$u_3(x,t) = \frac{-t^{3\alpha}}{\Gamma(3\alpha+1)} \sum_{n=1}^{\infty} \left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2} \right] \times \left(\frac{(2n-1)\pi}{2} \right)^6 \sin\left(\frac{(2n-1)\pi x}{2} \right), \quad (5.35)$$

$$u_4(x,t) = \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \sum_{n=1}^{\infty} \left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2} \right] \times \left(\frac{(2n-1)\pi}{2} \right)^8 \sin\left(\frac{(2n-1)\pi x}{2} \right), \quad (5.36)$$

and so on. Therefore, the solution is:

$$u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1}}{(2n-1)^2} \right] \times \sum_{k=0}^{\infty} \frac{\left(-\frac{(2n-1)^2\pi^2 t^\alpha}{4} \right)^k}{\Gamma(\alpha k+1)} \sin\left(\frac{(2n-1)\pi x}{2} \right), \quad (5.37)$$

which is the exact solution of the problem.

Example 3.

$$D_t^\beta u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < \beta < 2, \quad (5.38)$$

where $D_t^\beta(\bullet)$ is the Caputo derivative of order β.

With the initial conditions:

$$u(x,0) = \delta(x), \quad u_t(x,0) = 0. \quad (5.39)$$

Taking the Fourier transform of Equations (5.38) and (5.39), we obtain:

$$D_t^\beta \bar{u}(\omega,t) = -\omega^2 \bar{u}(\omega,t), \quad (5.40)$$

$$\bar{u}(\omega,0) = \frac{1}{\sqrt{2\pi}}, \quad (5.41)$$

where $\bar{u}(\omega,t) = 1/\sqrt{2\pi} \int_{-\infty}^{+\infty} e^{+i\omega x} u(x,t)dx, \omega \in R$.

According to HPM, we readily construct the homotopy:

$$D_t^\beta \bar{u}(\omega,t) = -p\omega^2 \bar{u}(\omega,t). \quad (5.42)$$

Assume the solution of equation in the form:

$$\bar{u} = \bar{u}_0 + p\bar{u}_1 + p^2\bar{u}_2 + p^3\bar{u}_3 + p^4\bar{u}_4 + \dots. \quad (5.43)$$

Substituting (5.43) into Equation (5.42) and collecting terms of the same power of p gives:

$$p^0 : D_t^\beta \bar{u}_0 = 0, \quad \bar{u}(\omega,0) = \bar{u}_0(\omega), \quad (5.44)$$

$$p^1 : D_t^\beta \bar{u}_1 = -\omega^2 \bar{u}_0, \quad \bar{u}_1(\omega,0) = 0, \quad (5.45)$$

$$p^2 : D_t^{\beta}\bar{u}_2 = -\omega^2\bar{u}_1, \quad \bar{u}_2(\omega, 0) = 0, \tag{5.46}$$

$$p^3 : D_t^{\beta}\bar{u}_3 = -\omega^2\bar{u}_2, \quad \bar{u}_3(\omega, 0) = 0, \tag{5.47}$$

$$\cdots$$

Solving the above equations, we obtain:

$$\bar{u}_0(\omega, t) = \frac{1}{\sqrt{2\pi}}, \tag{5.48}$$

$$\bar{u}_1(\omega, t) = \frac{-\omega^2}{\sqrt{2\pi}}\frac{t^{\beta}}{\Gamma(\beta+1)}, \tag{5.49}$$

$$\bar{u}_2(\omega, t) = \frac{\omega^4}{\sqrt{2\pi}}\frac{t^{2\beta}}{\Gamma(2\beta+1)}, \tag{5.50}$$

$$\bar{u}_3(\omega, t) = \frac{-\omega^6}{\sqrt{2\pi}}\frac{t^{3\beta}}{\Gamma(3\beta+1),} \tag{5.51}$$

$$\cdots$$

and so on.

Then setting $p = 1$ results in the approximate solution of the equation:

$$\bar{u} = \bar{u}_0 + p\bar{u}_1 + p^2\bar{u}_2 + p^3\bar{u}_3 + p^4\bar{u}_4 + \cdots, \tag{5.52}$$

$$\bar{u}(\omega, t) = \frac{1}{\sqrt{2\pi}}\left[1 - \frac{\omega^2 t^{\beta}}{\Gamma(\beta+1)} + \frac{\omega^4 t^{2\beta}}{\Gamma(2\beta+1)} - \frac{\omega^6 t^{3\beta}}{\Gamma(3\beta+1)} + \cdots\right], \tag{5.53}$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-\omega^2)^k t^{kp}}{\Gamma(k\beta+1)}. \tag{5.54}$$

Taking the inverse Fourier transform of the above equation we obtain the solution

$$u(x, t) = \frac{1}{2}t^{-\beta/2}M_{\beta/2}(|x|/t^{\beta/2}), \quad -\infty < x + \infty, \quad t \geq 0, \tag{5.55}$$

where $u(x, t) = 1/\sqrt{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega \mathrm{x}}\bar{u}(\omega, t)d\omega, x \in \mathrm{R}$ and

$$M_{\beta/2}(|x|/t^{\beta/2}) = \sum_{n=0}^{\infty}\frac{(|x|/t^{\beta/2})^2}{n!\Gamma\left[-\frac{n\beta}{2} + \left(1 - \frac{\beta}{2}\right)\right]}, \quad 0 < \frac{\beta}{2} < 1. \tag{5.56}$$

Here, $M_{\beta/2}$ denotes the so-called M function of order $\beta/2$, which is a special case of the Wright function (Mainardi and Pagnini, 2003).

Example 4. Let us consider (1 + 1)-dimensional nonlinear fractional equation:

$$D_t^\alpha u - \gamma^2 u_{xx} + c^2 u - \sigma u^3 = 0, \quad 1 < \alpha \le 2, \tag{5.57}$$

with the initial conditions:

$$u(x,0) = \varepsilon \cos kx, \quad u_t(x,0) = 0. \tag{5.58}$$

According to HPM, we readily construct the homotopy:

$$D_t^\alpha u = p[\gamma^2 u_{xx} - c^2 u + \sigma u^3]. \tag{5.59}$$

Substituting (5.6) into Equation (5.5) and collecting terms of the same power of p gives:

$$p^0 : D_t^\alpha u_0 = 0, \quad u(x,0) = u_0(x), \tag{5.60}$$

$$p^1 : D_t^\alpha u_1 = [\gamma^2 [u_0]_{xx} - c^2 u_0 + \sigma [u_0]^3], \tag{5.61}$$

$$p^2 : D_t^\alpha u_2 = [\gamma^2 [u_1]_{xx} - c^2 u_1 + \sigma [3[u_0]^2 u_1]], \tag{5.62}$$

$$\ldots$$

Solving the above equations, we obtain:

$$u_0(x,t) = \varepsilon \cos kx, \tag{5.63}$$

$$u_1(x,t) = \varepsilon \cos kx - \frac{\gamma^2 \varepsilon k^2 t^\alpha \cos(kx)}{\Gamma(\alpha+1)} - \frac{(c^2 \varepsilon \cos(kx) - \sigma\varepsilon^3 \cos^3(kx)) t^\alpha}{\Gamma(\alpha+1)}, \tag{5.64}$$

$$\begin{aligned} u_2(x,t) = {} & \varepsilon \cos kx - \frac{\gamma^2 \varepsilon k^2 t^\alpha \cos(kx)}{\Gamma(\alpha+1)} - \frac{(c^2 \varepsilon \cos(kx) - \sigma\varepsilon^3 \cos^3(kx)) t^\alpha}{\Gamma(\alpha+1)} \\ & + \frac{\varepsilon\gamma^4 k^4 \cos(kx) t^{2\alpha}}{\Gamma(2\alpha+1)} - \gamma^2 \left[-c^2 \varepsilon k^2 \cos(kx) + \frac{3k^2 \sigma \varepsilon^3}{4} \cos(kx) \right. \\ & \left. + \frac{9k^2 \sigma\varepsilon^3}{4} \cos(3kx) \right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{c^2 \varepsilon \gamma^2 k^2 \cos(kx) t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & + \frac{c^2 (c^2 \varepsilon \cos(kx) - \sigma\varepsilon^3 \cos^3(kx)) t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{3\sigma\varepsilon^3 \gamma^2 k^2 \cos^3(kx) t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & - \frac{3\sigma\varepsilon^2 \cos^2(kx)(c^2 \varepsilon \cos(kx) - \sigma\varepsilon^3 \cos^3(kx)) t^{2\alpha}}{\Gamma(2\alpha+1)} \end{aligned} \tag{5.65}$$

$$\ldots$$

and so on.

5.2 Applications of the VIM

Example 1. According to VIM, the iteration formula for Equations (5.1)-(5.4) is given by:

$$u_{k+1}(x,t) = u_k(x,t) - J_t^{\alpha}\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_k(x,t) - \frac{\partial^2}{\partial x^2}u_k(x,t)\right]. \tag{5.66}$$

By the above variational iteration formula, if we begin with:

$$u_0(x,t) = \sin(x), \tag{5.67}$$

we can obtain the following approximates:

$$u_1(x,t) = \sin x - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin(x), \tag{5.68}$$

$$u_2(x,t) = \sin x - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x), \tag{5.69}$$

$$u_3(x,t) = \sin x - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x) - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\sin(x), \tag{5.70}$$

$$\begin{aligned} u_4(x,t) = \sin x &- \frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x) - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\sin(x) \\ &+ \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\sin(x), \end{aligned} \tag{5.71}$$

...

and so on.

Therefore, the solution is:

$$u(x,t) = \sum_{r=0}^{\infty}\frac{(-t^{\alpha})^r}{\Gamma(r\alpha+1)}\sin(x), \tag{5.72}$$

which is the exact solution of the problem. The solution can be verified through substitution in equation.

Example 2. By the variational iteration formula (5.66) for Equations (5.19)-(5.24), if we begin with:

$$\begin{aligned} u_0 &= u(x,0) + tu_t(x,0) \\ &= \sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\sin\left(\frac{(2n-1)\pi x}{2}\right). \end{aligned} \tag{5.73}$$

We can obtain the following approximates:

$$u_1(x,t)=\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\sin\left(\frac{(2n-1)\pi x}{2}\right)-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\times\left(\frac{(2n-1)\pi}{2}\right)^2\sin\left(\frac{(2n-1)\pi x}{2}\right), \tag{5.74}$$

$$\begin{aligned}u_2(x,t)&=\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\sin\left(\frac{(2n-1)\pi x}{2}\right)-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\\&\times\left(\frac{(2n-1)\pi}{2}\right)^2\sin\left(\frac{(2n-1)\pi x}{2}\right)+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\\&\times\left(\frac{(2n-1)\pi}{2}\right)^4\sin\left(\frac{(2n-1)\pi x}{2}\right),\end{aligned} \tag{5.75}$$

$$\begin{aligned}u_3(x,t)&=\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\sin\left(\frac{(2n-1)\pi x}{2}\right)-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\\&\times\left(\frac{(2n-1)\pi}{2}\right)^2\sin\left(\frac{(2n-1)\pi x}{2}\right)+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\\&\times\left(\frac{(2n-1)\pi}{2}\right)^4\sin\left(\frac{(2n-1)\pi x}{2}\right)-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\\&\times\left(\frac{(2n-1)\pi}{2}\right)^6\sin\left(\frac{(2n-1)\pi x}{2}\right),\end{aligned} \tag{5.76}$$

$$\begin{aligned}u_4(x,t)&=\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\sin\left(\frac{(2n-1)\pi x}{2}\right)-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\\&\times\left(\frac{(2n-1)\pi}{2}\right)^2\sin\left(\frac{(2n-1)\pi x}{2}\right)+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\\&\times\left(\frac{(2n-1)\pi}{2}\right)^4\sin\left(\frac{(2n-1)\pi x}{2}\right)-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\\&\times\left(\frac{(2n-1)\pi}{2}\right)^6\sin\left(\frac{(2n-1)\pi x}{2}\right)+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\sum_{n=1}^{\infty}\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]\\&\times\left(\frac{(2n-1)\pi}{2}\right)^8\sin\left(\frac{(2n-1)\pi x}{2}\right),\end{aligned} \tag{5.77}$$

$\cdots$

and so on.

Therefore, the solution is:

$$u(x,t)=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}}{(2n-1)^2}\right]\times\sum_{k=0}^{\infty}\frac{\left(-\dfrac{(2n-1)^2\pi^2t^{\alpha}}{4}\right)^k}{\Gamma(\alpha k+1)}\sin\left(\frac{(2n-1)\pi x}{2}\right). \quad (5.78)$$

Example 3. According to VIM, the iteration formula for Equations (5.38)-(5.42) is given by:

$$\bar{u}_{k+1}(\omega,t)=\bar{u}_k(\omega,t)-J_t^{\alpha}\left[\frac{\partial^{\beta}}{\partial t^{\beta}}\bar{u}_k(\omega,t)+\omega^2\bar{u}_k(\omega,t)\right]. \quad (5.79)$$

By the above variational iteration formula, we begin with:

$$\bar{u}_0(\omega,t)=\frac{1}{\sqrt{2\pi}}, \quad (5.80)$$

and find:

$$\bar{u}_1(\omega,t)=\frac{1}{\sqrt{2\pi}}-\frac{\omega^2}{\sqrt{2\pi}}\frac{t^{\beta}}{\Gamma(\beta+1)}, \quad (5.81)$$

$$\bar{u}_2(\omega,t)=\frac{1}{\sqrt{2\pi}}-\frac{\omega^2}{\sqrt{2\pi}}\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{\omega^4}{\sqrt{2\pi}}\frac{t^{2\beta}}{\Gamma(2\beta+1)}, \quad (5.82)$$

$$\bar{u}_3(\omega,t)=\frac{1}{\sqrt{2\pi}}-\frac{\omega^2}{\sqrt{2\pi}}\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{\omega^4}{\sqrt{2\pi}}\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\frac{\omega^6}{\sqrt{2\pi}}\frac{t^{3\beta}}{\Gamma(3\beta+1),} \quad (5.83)$$

$\dots$

and so on.

$$\bar{u}(\omega,t)=\frac{1}{\sqrt{2\pi}}\left[1-\frac{\omega^2t^{\beta}}{\Gamma(\beta+1)}+\frac{\omega^4t^{2\beta}}{\Gamma(2\beta+1)}-\frac{\omega^6t^{3\beta}}{\Gamma(3\beta+1)}+\cdots\right], \quad (5.84)$$

$$=\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-\omega^2)^k t^{kp}}{\Gamma(k\beta+1)}. \quad (5.85)$$

Taking the inverse Fourier transform of the above equation, we obtain the solution:

$$u(x,t)=\frac{1}{2}t^{-\beta/2}M_{\beta/2}(|x|/t^{\beta/2}), \quad -\infty<x+\infty, \quad t\geq 0. \quad (5.86)$$

Example 4. According to VIM, the iteration formula for Equations (5.57)-(5.58) is given by:

$$u_{k+1}(x,t) = u_k(x,t) - J_t^{\alpha}\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_k(x,t) - \gamma^2\frac{\partial^2}{\partial x^2}u_k(x,t) + c^2u_k(x,t) - \sigma u_k(x,t)^3\right]. \tag{5.87}$$

By the above variational iteration formula, we begin with:

$$u_0(x,t) = \varepsilon \cos kx, \tag{5.88}$$

and find:

$$u_1(x,t) = \varepsilon \cos kx - \frac{\gamma^2\varepsilon k^2 t^{\alpha}\cos(kx)}{\Gamma(\alpha+1)} - \frac{(c^2\varepsilon\cos(kx) - \sigma\varepsilon^3\cos^3(kx))t^{\alpha}}{\Gamma(\alpha+1)}, \tag{5.89}$$

$$\begin{aligned} u_2(x,t) = {} & \varepsilon \cos kx - \frac{\gamma^2\varepsilon k^2 t^{\alpha}\cos(kx)}{\Gamma(\alpha+1)} - \frac{(c^2\varepsilon\cos(kx) - \sigma\varepsilon^3\cos^3(kx))t^{\alpha}}{\Gamma(\alpha+1)} \\ & + \frac{\varepsilon\gamma^4 k^4\cos(kx)t^{2\alpha}}{\Gamma(2\alpha+1)} - \gamma^2\left[-c^2\varepsilon k^2\cos(kx) + \frac{3k^2\sigma\varepsilon^3}{4}\cos(kx)\right. \\ & \left. + \frac{9k^2\sigma\varepsilon^3}{4}\cos(3kx)\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{c^2\varepsilon\gamma^2 k^2\cos(kx)t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & + \frac{c^2(c^2\varepsilon\cos(kx) - \sigma\varepsilon^3\cos^3(kx))t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{3\sigma\varepsilon^3\gamma^2 k^2\cos^3(kx)t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & - \frac{3\sigma\varepsilon^2\cos^2(kx)(c^2\varepsilon\cos(kx) - \sigma\varepsilon^3\cos^3(kx))t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{5.90}$$

...
and so on.

6. Conclusion

In this study, we demonstrate that present methods are also well suited to solving time-fractional diffusion-wave and nonlinear fractional equations. The HPM and VIM are straightforward, without restrictive assumptions, and the components of the series solution can be easily computed using any mathematical symbolic package. Moreover, these methods do not change the problem into a convenient one for the use of linear theory. They, therefore, provide more realistic series solutions that generally converge very rapidly in real physical problems. When solutions are computed numerically, the rapid convergence is obvious. Moreover, no linearization or perturbation is required. They can avoid the difficulty of finding the inverse of the Laplace transform and can reduce the labor of perturbation method. Furthermore, as the HPM and VIM do require discretization of the variables, i.e. time and space, it is not affected by computational round off errors and one is not faced with the necessity of large computer memory and time. Consequently, the computational load will be reduced.

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